Pergamon

Int. J. Solids Structures Vol. 33, No. 8, pp. 1083–1103, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00074–7

DAMAGE, GRADIENT OF DAMAGE AND PRINCIPLE OF VIRTUAL POWER

MICHEL FRÉMOND and BOUMEDIENE NEDJAR
Laboratoire des matériaux et des structures du génie civil, Laboratoire mixte LCPC/CNRS, UMR 113, 2, allée Kepler, Cité Descartes, 77420, Champs sur Marne, France

(*Received* 11 *October* 1994; *in revised form* 13 *March* 1995)

Abstract—A theory of continuum damage mechanics is developed within the framework of the principle of virtual power. Because the damage in a solid results from microscopic movements, we decided to include the power of these microscopic movements in the power of the internal forces. The microscopic velocities are related to the damage rate. The power of the internal forces we choose depends on the damage velocity and its gradient to take into account the interactions. Models issued from this theory are presented. They are coherent from the mechanical and mathematical points of view. The numerical computations show no mesh sensitivity. They describe with good agreement the main experimental properties. Concrete is chosen as an example to illustrate the theory. A model using two damage quantities is also presented. It yields the correct description of the unilateral phenomenon observed in concrete. Finally, an extension of the existing models is proposed to describe the fatigue damage behaviour.

1. INTRODUCTION

Damage of materials has been modelled for a long time by damage quantities within the framework of continuum mechanics. They are internal quantities which appear in the expression of the free energy of the material. The many possible expressions for the free energy yield numerous and versatile constitutive laws. The constitutive laws are coupled with the balance law resulting from the principle of virtual power to give predictive theories [see for instance Germain *et al.* (1983) and Lemaitre and Chaboche (1988)].

The principle of virtual power is often thought to be very fixed. Nevertheless, it is possible to modify it or better adapt it to the problem under consideration. Damage, for instance damage of concrete, results from microscopic movements. Our basic idea is that the power of these microscopic movements must be accounted for in a predictive theory. Thus, we decide to modify the expression of the power of the internal forces. We assume that this power also depends on the damage rate, which is clearly related to the microscopic movements. Furthermore, we assume that it also depends on the gradient of the damage rate to account for microscopic interactions. The consequences of this assumption and a careful treatment of the fact that the damage quantities are proportions (i.e. quantities with values between 0 and 1) are given in Section 2. The basic constitutive laws are derived from that analysis. The main results have been briefly presented in Frémond and Nedjar (1993a, b).

The models issued from this formulation are free of spurious mesh sensitivity and are able, when compared to experimental results, to predict correctly the behaviour of concrete structures. Also, accounting for the gradient of damage leads to good predictions of the structural size effect, which is particulary important in civil engineering.

The paper is organized as follows: the next section is devoted to the formulation of the damage theory founded on the principle of virtual powers. The constitutive laws are given following the principles of continuum thermodynamics (Germain *et al.*, 1983; Frémond, 1990). In Sections 3 and 4, damage models are developed within the framework of the previous theory. The unilateral phenomenon, which means the macroscopic restoration of stiffness when going from tension to compression, is described by using two damage quantities.

In Sections 5 and 6, numerical examples on concrete structures are given where some possibilities of the models are emphasized: the lack of mesh sensitivity and the structural size effect. Finally, in Section 7, an extension of an existing model (of Section 3) is proposed to describe the fatigue damage behaviour.

2. BALANCE LAWS AND CONSTITUTIVE LAWS FOR DAMAGE

We consider a solid, for instance a piece of concrete, and study its damage. Within the framework of continuum mechanics, we want to describe at the macroscopic level the effects of microfractures and microcavities which result in a decrease of the material stiffness. Let the scalar $\beta(\mathbf{x}, t)$ be the macroscopic damage quantity with value 1 when the material is undamaged and value 0 when it is completely damaged. There exist classical theories describing the damage of materials. They are well established from the mechanical point of view. The finite element approximations of the resulting set of partial differential equations have an unusual property: the numerical results depend heavily on the finite element mesh. This property can be thought to be unacceptable or acceptable due to the so-called size effect. In this paper we describe a predictive theory which is not mesh dependent. It is also coherent from the mechanical and mathematical points of view.

The basic idea of the theory is to modify the power of the interior forces (Frémond, 1987). Within the solid, there exist microscopic movements which produce damage. We think that the power of these microscopic movements must be taken into account in the power of the internal forces. Thus, we choose the power of the internal forces to depend, besides on the strain rates $\mathbf{D}(\mathbf{u})$ ($\mathbf{u}$ is the macroscopic velocity), also on $\mathrm{d}\beta/\mathrm{d}t$ and grad $\mathrm{d}\beta/\mathrm{d}t$. These latter quantities are clearly related to the microscopic movements. The gradient of damage is introduced to take into account the influence of damage at a material point on the damage of its neighbourhood.

The principle of virtual power gives a new equation which describes the evolution of the damage quantity β. It is natural to assume that the free energy Ψ is a function of the deformations, β and grad β. For the sake of simplicity we assume that there is only dissipation with respect to $\mathrm{d}\beta/\mathrm{d}t$. We also assume that the temperature is constant.

2.1. *Principle of virtual power: equations of movement*

We choose the power of the internal forces which takes into account the microscopic movements in a domain Ω of the solid as:

$$P_{\mathrm{i}}\left(\mathbf{u}, \frac{\mathrm{d}\beta}{\mathrm{d}t}\right) = -\int_{\Omega} \sigma : \mathbf{D}(\mathbf{u})\,\mathrm{d}\Omega - \int_{\Omega}\left(B\frac{\mathrm{d}\beta}{\mathrm{d}t} + \mathbf{H}\cdot \mathrm{grad}\frac{\mathrm{d}\beta}{\mathrm{d}t}\right)\mathrm{d}\Omega,$$

where σ is the stress tensor. Two new non-classical quantities appear: B, the internal work of damage, and $\mathbf{H}$, the flux vector of internal work of damage.

The axiom of the principle of virtual power is satisfied: $P_{\mathrm{i}} = 0$ for any rigid body velocity ($\mathrm{d}\beta/\mathrm{d}t = 0$ in such a movement because the distance of material points remains constant).

We choose the power of the external forces as:

$$P_{\mathrm{e}}\left(\mathbf{u}, \frac{\mathrm{d}\beta}{\mathrm{d}t}\right) = \int_{\Omega} \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\Omega + \int_{\partial\Omega} \mathbf{F}\cdot\mathbf{u}\,\mathrm{d}\Gamma + \int_{\Omega} A\frac{\mathrm{d}\beta}{\mathrm{d}t}\mathrm{d}\Omega + \int_{\partial\Omega} b\frac{\mathrm{d}\beta}{\mathrm{d}t}\mathrm{d}\Gamma,$$

where $\mathbf{f}$ are the volumetric external forces, $\mathbf{F}$ the surfacic external forces, A and b are respectively the volumetric and surface external sources of damage work. A source of damage work A or b can be produced by chemical (or in some cases electrical) actions which break the links inside a material, concrete for instance, without macroscopic deformations. One can think, for instance, of the so-called alkali aggregate reaction which damages concrete.

We choose the power of the acceleration forces as:

$$P_a\left(\mathbf{u}, \frac{d\beta}{dt}\right) = \int_\Omega \rho \frac{d\mathbf{u}}{dt} \cdot \mathbf{u} \, d\Omega + \int_\Omega \rho \frac{d^2\beta}{dt^2} \frac{d\beta}{dt} d\Gamma,$$

where $\mathbf{u}$ is the macroscopic velocity and ρ the density. The quantity $\rho(d^2\beta/dt^2)$ stands for the acceleration forces of the microscopic links; ρ is proportionnal to their mass.

For quasi-static evolutions, the principle of virtual power,

$$\forall \Omega, \quad \forall \mathbf{v}, \quad \forall \gamma, \quad P_i(\mathbf{v}, \gamma) + P_e(\mathbf{v}, \gamma) = 0$$

($\mathbf{v}$ and γ are virtual velocities), gives two sets of movement equations:

$$\operatorname{div} \sigma + \mathbf{f} = 0, \quad \text{in } \Omega; \qquad \sigma \cdot \mathbf{n} = \mathbf{F}, \quad \text{in } \partial\Omega; \tag{1}$$

$$\operatorname{div} \mathbf{H} - B + A = 0, \quad \text{in } \Omega; \qquad \mathbf{H} \cdot \mathbf{n} = b, \quad \text{in } \partial\Omega, \tag{2}$$

where $\mathbf{n}$ is the outwards unit normal vector to Ω. Equation (2) is new and non-classical.

2.2. *Internal constraint on the damage quantity: constitutive laws*

The value of the damage quantity β is between 0 and 1 (β is often thought to be the volumetric proportion of microvoids or the quotient of the modulus of the damaged material by the modulus of the undamaged material):

$$0 \leqslant \beta \leqslant 1. \tag{3}$$

We take the internal constraint (3) on the damage quantity to be a physical property. Being a physical property of a state quantity, it must be taken into account by the functions which describe the whole physical properties, i.e. either the free energy Ψ or the dissipative forces which can be defined by a pseudo-potential of dissipation Φ. We choose the free energy because the free energy describes properties related to the state and the dissipative forces properties related to the velocities. For the sake of simplicity, we make the small perturbations assumption and let ε be the small deformations. We choose

$$\Psi = \Psi(\varepsilon, \beta, \operatorname{grad} \beta) = \Psi_1(\varepsilon, \beta, \operatorname{grad} \beta) + I(\beta),$$

where Ψ_1 is a smooth function and I is the indicator function of the set [0, 1] (Moreau, 1966) ($I(\gamma) = 0$, if $0 \leqslant \gamma \leqslant 1$ and $I(\gamma) = +\infty$, if $\gamma \notin [0, 1]$). Thus, the free energy has a physical value for any actual or physical value of β. The free energy is equal to $+\infty$ for any value of β which is physically impossible, for instance $\beta > 1$. Due to the expression of the power of the interior forces, it is natural to assume that the free energy depends on the gradient of the state quantity β. This choice, based on the expression of the power of the interior forces, has already been made to describe adhesion (Frémond, 1987). The gradient of internal quantities has also been used in another general setting (Maugin, 1990; Costa Mattos *et al.*, 1992).

The computation of the derivatives of the free energy in an actual evolution, i.e. in an evolution such that $0 \leqslant \beta(\mathbf{x}, t) \leqslant 1$ for any point $\mathbf{x}$ and any time t, allows us to define the reversible or non-dissipative forces related to ε, β and grad β:

$$\sigma^r(\mathbf{x}, t) = \frac{\partial \Psi_1}{\partial \varepsilon}(\mathbf{x}, t), \quad B^r(\mathbf{x}, t) = \frac{\partial \Psi_1}{\partial \beta}(\mathbf{x}, t), \quad \mathbf{H}^r(\mathbf{x}, t) = \frac{\partial \Psi_1}{\partial \operatorname{grad} \beta}(\mathbf{x}, t). \tag{4}$$

The internal constraint (3) is taken into account by introducing a reaction B^{reac} which is defined by assuming that there exists a function $B^{reac}(\mathbf{x}, t)$ such that

$$B^{reac}(\mathbf{x}, t) \in \partial I(\beta(\mathbf{x}, t)), \tag{5}$$

where ∂I is the subdifferential or the generalized derivative of the indicator function I (Moreau, 1966) ($\partial I(\beta) = \{0\}$ if $0 < \beta < 1$, $\partial I(0) = \mathbf{R}^-$, $\partial I(1) = \mathbf{R}^+$, $\partial I(\beta) = \varnothing$, if $\beta \notin [0, 1]$).

Let us note that relation (5) implies that the subdifferential $\partial I(\beta)$ is not empty, and thus that the internal constraint (3) is satisfied. One can also say that relation (5) has two meanings, first that the internal constraint is satisfied, second that there exists a reaction to the internal constraint which is zero for $0 < \beta < 1$, positive for $\beta = 1$ and negative for $\beta = 0$. Let us also note that the sum of the reaction B^{reac} and of the reversible force B^r, $(B^{reac} + B^r)$ is the generalized derivative of the free energy Ψ with respect to β; B^r is the smooth part and B^{reac} is the non-smooth part of the derivative. If the indicator function I is approximated by a smooth function, like in Brauner *et al.* (1986), B^{reac} is approximated by a classical derivative and there is no more difference between the smooth part B^r and the non-smooth part B^{reac}. In our point of view, the non-smooth mechanics point of view, the free energy is Ψ and the non-dissipative force associated to β is $B^r + B^{reac} \in \partial\Psi$.

For the sake of simplicity we assume that there is no dissipation with respect to the small deformations ε (i.e. the material is elastic) and with respect to the gradient of the damage quantity. Thus, we assume that there are only dissipative phenomena, viscous phenomena for instance, for the damage quantity β.

To define the dissipative force associated to β, we assume that there exists a function

$$B^i\left(\mathbf{x}, t, \varepsilon, \beta, \operatorname{grad}\beta, \frac{d\varepsilon}{dt}, \frac{d\beta}{dt}, \frac{d\operatorname{grad}\beta}{dt}\right),$$

such that:

$$\forall\mathbf{x},\quad \forall t,\quad \forall\varepsilon\in\mathbb{S},\quad \forall\beta\in\mathbf{R},\quad \forall\operatorname{grad}\beta\in\mathbf{R}^3,\quad \forall\frac{d\varepsilon}{dt}\in\mathbb{S},\quad \forall b\in\mathbf{R},\quad \forall\frac{d\operatorname{grad}\beta}{dt}\in\mathbf{R}^3,$$

$$B^i\left(\mathbf{x}, t, \varepsilon, \beta, \operatorname{grad}\beta, \frac{d\varepsilon}{dt}, b, \frac{d\operatorname{grad}\beta}{dt}\right) b \geqslant 0, \tag{6}$$

where $\mathbb{S}$ is the set of the 3×3 symmetric matrices. We let $E = (\varepsilon, \beta, \operatorname{grad}\beta)$.

The constitutive laws we choose are:

$$\sigma = \sigma^r,\quad \mathbf{H} = \mathbf{H}^r,\quad B = B^r + B^{reac} + B^i, \tag{7}$$

where the last constitutive law means that

$$B(\mathbf{x}, t) = B^r(\mathbf{x}, t) + B^{reac}(\mathbf{x}, t) + B^i\left(\mathbf{x}, t, E(\mathbf{x}, t), \frac{d\varepsilon}{dt}(\mathbf{x}, t), \frac{d\beta}{dt}(\mathbf{x}, t), \frac{d\operatorname{grad}\beta}{dt}(\mathbf{x}, t)\right).$$

The equations describing the evolution of a piece of material are (1), (2) and (7), completed by initial and boundary conditions. Before we investigate them, let us prove that our choice is coherent from a mechanical point of view. The only thing we have to prove is that the constitutive laws (7) are such that the Clausius–Duhem inequality is satisfied. Because we have assumed the temperature to be constant, the Clausius–Duhem inequality is:

$$\frac{d\Psi_1}{dt} \leqslant \sigma : \frac{d\varepsilon}{dt} + B\frac{d\beta}{dt} + \mathbf{H} \cdot \frac{d\,\mathrm{grad}\,\beta}{dt}, \tag{8}$$

for any actual evolution, i.e. for any actual velocities $d\varepsilon/dt$, $d\beta/dt$, $d\,\mathbf{grad}\,\beta/dt$ such that condition (3) is satisfied.

Before we go on, let us remark that we have to define carefully the time derivative $d\beta/dt$. Because of the inequality (3) the time derivative of β is not continuous; thus, we have to choose between the right derivative,

$$\lim \frac{\beta(t+\Delta t)-\beta(t)}{\Delta t} = \frac{d^r\beta}{dt}, \quad \Delta t \to 0(\Delta t > 0)$$

which depends on the future evolution of the material, and the left derivative,

$$\lim \frac{\beta(t)-\beta(t-\Delta t)}{\Delta t} = \frac{d^l\beta}{dt}, \quad \Delta t \to 0(\Delta t > 0)$$

which depends on the past evolution of the material.

We know that the constitutive laws are objective relations, i.e. relations which are computed at time t with the available information given by the history of the material, i.e. by its past evolution. It results that in the constitutive laws the derivatives with respect to the time are left derivatives (Frémond, 1990). Thus, we decide that all the time derivatives we consider or have considered are left derivatives:

$$\frac{d\pi}{dt} = \frac{d^l\pi}{dt}, \quad \text{for any quantity } \pi.$$

Let us prove the following result.

Theorem. Let us assume that the function $\Psi_1(\varepsilon, \beta, \mathrm{grad}\,\beta)$ is smooth and that the relations (4) are satisfied. Then the Clausius–Duhem inequality is satisfied by the constitutive laws (7).

Proof. Let us prove the following proposition before we prove the theorem.

Proposition (Frémond, 1990). In any actual evolution, i.e. in any evolution such that $\forall(\mathbf{x}, t)$, $0 \leqslant \beta(\mathbf{x}, t) \leqslant 1$, we have

$$\forall(\mathbf{x}, t), \quad \forall A \in \partial I(\beta(\mathbf{x}, t)), \quad A\frac{d\beta}{dt}(\mathbf{x}, t) \geqslant 0. \tag{9}$$

Proof of the proposition. Let $\Delta t > 0$ be a time increment. Because $0 \leqslant \beta(\mathbf{x}, t) \leqslant 1$, the indicator function I is subdifferentiable at the point $\beta(\mathbf{x}, t)$ (Moreau, 1966) and

$$\forall A \in \partial I(\beta(\mathbf{x}, t)), \quad I(\beta(\mathbf{x}, t-\Delta t)) \geqslant I(\beta(\mathbf{x}, t)) + A(\beta(\mathbf{x}, t-\Delta t) - \beta(\mathbf{x}, t))$$

or because $0 \leqslant \beta(\mathbf{x}, t-\Delta t) \leqslant 1$,

$$\forall A \in \partial I(\beta(\mathbf{x}, t)), \quad 0 \geqslant A(\beta(\mathbf{x}, t-\Delta t) - \beta(\mathbf{x}, t)).$$

By dividing this relation by $\Delta t > 0$, we get

$$\forall A \in \partial I(\beta(\mathbf{x}, t)), \quad A\frac{(\beta(\mathbf{x}, t) - \beta(\mathbf{x}, t-\Delta t))}{\Delta t} \geqslant 0.$$

By letting Δt tend to 0, we get the relation (9).

Proof of the theorem. Because the function $\psi_1(\varepsilon, \beta, \mathrm{grad}\,\beta)$ is smooth, we have

$$\frac{d\Psi_1}{dt} = \frac{\partial\Psi_1}{\partial\varepsilon}:\frac{d\varepsilon}{dt} + \frac{\partial\Psi_1}{\partial\beta}\frac{d\beta}{dt} + \frac{\partial\Psi_1}{\partial\,\mathrm{grad}\,\beta}\cdot\frac{d\,\mathrm{grad}\,\beta}{dt} = \sigma^r:\frac{d\varepsilon}{dt} + B^r\frac{d\beta}{dt} + \mathbf{H}^r\cdot\frac{d\,\mathrm{grad}\,\beta}{dt}.$$

Due to the proposition we have

$$0 \leqslant B^{\mathrm{reac}}(\mathbf{x},t)\frac{d\beta}{dt}(\mathbf{x},t).$$

The relation (6) gives

$$0 \leqslant B^{\mathrm{i}}\left(\mathbf{x},t,E,\frac{d\varepsilon}{dt},\frac{d\beta}{dt},\frac{d\,\mathrm{grad}\,\beta}{dt}\right)\frac{d\beta}{dt}.$$

By adding the last three relations, we get

$$\frac{d\Psi_1}{dt} \leqslant \sigma^r:\frac{d\varepsilon}{dt} + (B^r + B^{\mathrm{reac}} + B^{\mathrm{i}})\frac{d\beta}{dt} + \mathbf{H}^r\cdot\frac{d\,\mathrm{grad}\,\beta}{dt},$$

and by using the constitutive laws (7), we get the Clausius–Duhem inequality (8). That ends the proof of the theorem. ■

A very productive and elegant way to define dissipative forces is to assume that there exists a pseudo-potential of dissipation, as intoduced by Moreau (1970) [see also Germain *et al.* (1983)]. A pseudo-potential of dissipation is a positive, convex and sub-differentiable function of b, $\Phi(\mathbf{x},t,E,d\varepsilon/dt,b,d\,\mathrm{grad}\,\beta/dt)$ with value 0 for $b = 0$.

We define the dissipative force B^{i} satisfying condition (6) by

$$B^{\mathrm{i}}\left(\mathbf{x},t,E,\frac{d\varepsilon}{dt},\frac{d\beta}{dt},\frac{d\,\mathrm{grad}\,\beta}{dt}\right) \in \partial\Phi\left(\mathbf{x},t,E,\frac{d\varepsilon}{dt},\frac{d\beta}{dt},\frac{d\,\mathrm{grad}\,\beta}{dt}\right), \tag{10}$$

where $\partial\Phi(\mathbf{x},t,E,d\varepsilon/dt,d\beta/dt,d\,\mathrm{grad}\,\beta/dt)$ is the sub-differential of Φ with respect to b for $b = d\beta/dt$.

In the following sections we apply these basic results to practical situations: the damage of concrete and the damage of some composite materials. The different theories we describe are defined by different free energies Ψ and different pseudo-potentials of dissipation Φ, and also by choosing one or two damage quantities.

3. A MODEL WITH ONE DAMAGE QUANTITY

It is mainly observed that damage is produced by extensions into the material during loading when a certain threshold is achieved. Such situations are observed in the experimental tests on materials like concrete (Mazars and Bazant, 1988) and ceramic–ceramic composites (Ladevèze, 1983; Nedjar, 1993).

To be in agreement with these observations, and in the framework of the formulation of the damage theory described in Section 2, we are led to choose the free energy and the pseudo-potential of dissipation as

$$\Psi_1 = \tfrac{1}{2}\beta\{2\mu\,\mathrm{tr}\,[\varepsilon\cdot\varepsilon] + \lambda(\mathrm{tr}\,[\varepsilon])^2\} + W(1-\beta) - M\{\log(|\beta|) - \beta + 1\} + \frac{k}{2}(\mathrm{grad}\,\beta)^2, \tag{11}$$

$$\Phi = \tfrac{1}{2}c\dot{\beta}^2 - \tfrac{1}{2}\dot{\beta}\{2\mu\,\mathrm{tr}\,[\varepsilon^-\cdot\varepsilon^-] + \lambda(\langle\mathrm{tr}\,[\varepsilon]\rangle^-)^2\} + I_-(\dot{\beta}), \tag{12}$$

with the notation $\dot{\beta} = d\beta/dt$, and where λ and μ are the Lamé parameters. The first term of Ψ_1 is a quadratic function with respect to the strain tensor and a linear function with respect to the damage quantity. It constitutes the simplest model where the damage is coupled with elasticity. The quantity W is the initial damage threshold, expressed here in terms of volumetric energy. It is equivalent to the initial threshold expressed in terms of damage force (usually denoted by Y_0) conjugated to the damage quantity in the models issued from the theory of generalized standard materials (Pijaudier-Cabot and Bazant, 1987; Lemaitre and Chaboche, 1988). The quantity M is the factor of displacement of this threshold, c is the viscosity parameter of damage and k measures the influence of the damage at a material point on the damage of its neighbourhood.

The expression of the pseudo-potential of dissipation in eqn (12) is chosen in such a way that damage results only from extensions, as will be seen in the following.

The function I_- in eqn (12) is the indicator function of the interval $]-\infty, 0]$ ($I_-(\gamma) = 0$, if $\gamma \leqslant 0$ and $I_-(\gamma) = +\infty$, if $\gamma \notin]-\infty, 0]$). The effect of this indicator function is to make it compulsory for $\dot{\beta}$ to be negative. The notations $\langle\cdot\rangle^+$ and $\langle\cdot\rangle^-$ are respectively the positive part and the negative part of the scalar quantity $\langle\cdot\rangle$: $\langle\cdot\rangle^+ = \mathrm{Sup}\{0, \langle\cdot\rangle\}$ and $\langle\cdot\rangle^- = \mathrm{Sup}\{0, -\langle\cdot\rangle\}$. The positive part and the negative part of the strain tensor are obtained after diagonalization. One has the following useful properties (Ladevèze, 1983):

$$\langle \mathrm{tr}\,[\varepsilon]\rangle = \langle \mathrm{tr}\,[\varepsilon]\rangle^+ - \langle \mathrm{tr}\,[\varepsilon]\rangle^- \quad \text{and} \quad \langle \mathrm{tr}\,[\varepsilon]\rangle^+ \langle \mathrm{tr}\,[\varepsilon]\rangle^- = 0,$$

$$\varepsilon = \varepsilon^+ - \varepsilon^- \quad \text{and} \quad \mathrm{tr}\,[\varepsilon^+ \cdot \varepsilon^-] = 0,$$

$$\frac{1}{2}\frac{\partial\, \mathrm{tr}\,[\varepsilon^+ \cdot \varepsilon^+]}{\partial \varepsilon} = \varepsilon^+, \quad \frac{1}{2}\frac{\partial (\langle \mathrm{tr}\,[\varepsilon]\rangle^+)^2}{\partial \varepsilon} = \langle \mathrm{tr}\,[\varepsilon]\rangle^+ \mathbf{I}_d, \tag{13}$$

where $\mathbf{I}_d$ is the identity second order tensor.

With this choice, the constitutive relations (7) are:

$$\sigma = \beta\{2\mu\varepsilon + \lambda(\mathrm{tr}\,[\varepsilon])\mathbf{I}_d\}, \quad \mathbf{H} = k\,\mathrm{grad}\,\beta, \quad B = \frac{\partial \Psi_1}{\partial \beta} + \frac{\partial \Phi}{\partial \dot{\beta}} + B^{\mathrm{reac}}, \tag{14}$$

where the derivative of Ψ_1 and the generalized derivative of Φ (Moreau, 1966; Frémond, 1987, 1990) are:

$$\frac{\partial \Psi_1}{\partial \beta} = \tfrac{1}{2}\{2\mu\, \mathrm{tr}\,[\varepsilon \cdot \varepsilon] + \lambda(\mathrm{tr}\,\varepsilon)^2\} - W - M\left(\frac{1-\beta}{\beta}\right)$$

and

$$\frac{\partial \Phi}{\partial \dot{\beta}} \in c\dot{\beta} - \tfrac{1}{2}\{2\mu\, \mathrm{tr}\,[\varepsilon^- \cdot \varepsilon^-] + \lambda((\langle \mathrm{tr}\,\varepsilon\rangle^-)^2\} + \partial I(\dot{\beta}),$$

with $\partial I_-(x) = \{0\}$, if $x < 0$ and $\partial I_-(0) = [0, +\infty[$. One can see that the first equation in (14) relating the stress and the strain is the simplest relation where coupling of damage with elastic behaviour occurs.

The equations of evolution are obtained by replacing eqns (11), (12) and (14) in eqns (1) and (2). We then get:

$$\mathrm{div}\,(\beta\{2\mu\varepsilon + \lambda(\mathrm{tr}\,[\varepsilon])\mathbf{I}_d\}) + \mathbf{f} = 0, \quad \text{in } \Omega, \tag{15}$$

$$\sigma \cdot \mathbf{n} = \mathbf{F}, \quad \text{in } \partial\Omega,$$

$$c\dot{\beta} - k\Delta\beta + \partial I(\beta) + \partial I_-(\dot{\beta}) \ni -\tfrac{1}{2}\{2\mu\, \mathrm{tr}\,[\varepsilon^+ \cdot \varepsilon^+] + \lambda((\langle \mathrm{tr}\,[\varepsilon]\rangle^+)^2\} + W + M\left(\frac{1-\beta}{\beta}\right), \quad \text{in } \Omega,$$

$$k\frac{\partial \beta}{\partial \mathbf{n}} = 0, \quad \text{in } \partial\Omega, \quad \beta(x, 0) = \beta_0(x), \quad \text{in } \Omega, \tag{16a,b}$$

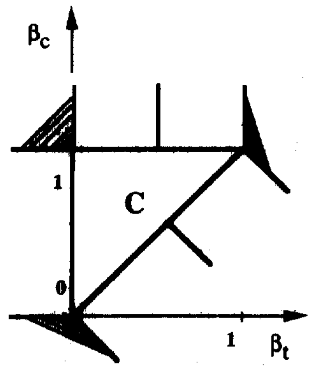


Fig. 1. Generalized derivatives of the indicator function I_C of the triangle C.

where $\Delta\beta$ is the Laplacian of β. The function β_0 is the initial value of the damage in Ω, with $\beta_0(x) = 1$ when the structure is initially undamaged.

Equations (16) are the equations of evolution of damage in the domain Ω. The elements $\partial I(\beta)$ and $\partial I_-(\dot{\beta})$ are sets of reactions which force β to remain between 0 and 1 and $\dot{\beta}$ to be negative. In eqn (16a), the source of damage on the right-hand-side is a strain energy produced by extensions. This agrees with the experimental observations mentioned above. It is also important to note that this equation shows different behaviours in tension and compression, due to Poisson's ratio, as already checked in Frémond and Nedjar (1993a) and Nedjar (1993). The threshold of damage is larger in compression than in tension. This is again in agreement with experimental tests on concrete structures. This model is sufficient to describe the damage phenomena during multiaxial loadings (loading–unloading without changing the sign of the external actions).

Examples using this model are given in Sections 5 and 6, where we check the lack of mesh sensitivity of the numerical results given by the finite element discretization and study the size effect in the structures modelled by this predictive theory of damage.

4. A TWO-DAMAGE-QUANTITIES MODEL AND THE UNILATERAL PHENOMENON: APPLICATION TO CONCRETE

When damage is produced, microcracks appear in the zones where extensions exist. When changing the sign of the principal deformations, by changing the sign of the loading, these microcracks close. At the macroscopic level, the initial stiffness is then recovered: this is a unilateral phenomenon (Ladevèze, 1983; Chaboche, 1992; Pijaudier-Cabot *et al.*, 1994). This unilateral phenomenon is not described by the previous model. To take it into account, this model is completed by introducing two damage quantities instead of a single one. As in Frémond and Nedjar (1993a, b), they are denoted as β_t for the extension and β_c for the contraction.

The principle of virtual powers discribed in Section 2 is then written with these two damage quantities. In this case, the internal constraint on the damage quantities (3) is replaced by

$$(\beta_t, \beta_c) \in C = \{(x, y), x \in [0, 1]; y \in [0, 1], x \leqslant y\}. \tag{17}$$

This constraint means that (β_t, β_c) remains in the triangle C. Therefore, we have $\beta_t \leqslant \beta_c$. This means that damage in compression produces damage in tension. However, the reverse is not true. The internal constraint results in two reactions B_t^{reac} and B_c^{reac} which are defined by

$$(B_t^{reac}(\mathbf{x}, t), B_c^{reac}(\mathbf{x}, t)) \in \partial I_C(\beta_t(\mathbf{x}, t), \beta_c(\mathbf{x}, t)), \tag{18}$$

where ∂I_C is the subdifferential or the set of the generalized derivatives of the indicator function I_C of C (see Fig. 1). We have:

$$B_t^{reac} = 0 \text{ and } B_c^{reac} = 0, \qquad \text{if } (\beta_t, \beta_c) \text{ is in the interior of } C,$$

$$B_t^{reac} \in]-\infty, 0] \text{ and } B_c^{reac} = 0, \qquad \text{if } \beta_t = 0 \text{ and } \beta_c \in]0, 1[,$$

$$B_t^{reac} = 0 \text{ and } B_c^{reac} \in [0, +\infty[, \qquad \text{if } \beta_t \in]0, 1[\text{ and } \beta_c = 1,$$

$$B_t^{reac} = -B_c^{reac} \in [0, +\infty[, \qquad \text{if } \beta_t = \beta_c \text{ and } \beta_t \in]0, 1[,$$

$$(B_t^{reac}, B_c^{reac}) \in \{\text{normal cone to } C \text{ at } (0,0)\}, \qquad \text{for } (\beta_t, \beta_c) = (0,0),$$

$$(B_t^{reac}, B_c^{reac}) \in \{\text{normal cone to } C \text{ at } (0,1)\}, \qquad \text{for } (\beta_t, \beta_c) = (0,1),$$

$$(B_t^{reac}, B_c^{reac}) \in \{\text{normal cone to } C \text{ at } (1,1)\}, \qquad \text{for } (\beta_t, \beta_c) = (1,1).$$

The free energy and the pseudo-potential of dissipation we choose are

$$\begin{aligned}\Psi_1 &= \Psi_1(\varepsilon, \beta_t, \beta_c, \operatorname{grad}\beta_t, \operatorname{grad}\beta_c) \\ &= \tfrac{1}{2}\{\beta_t\{2\mu \operatorname{tr}[\varepsilon^+ \cdot \varepsilon^+] + \lambda((\operatorname{tr}[\varepsilon])^+)^2\} + \beta_c\{2\mu \operatorname{tr}[\varepsilon^- \cdot \varepsilon^-] + \lambda((\operatorname{tr}[\varepsilon])^-)^2\}\} \\ &\quad + W_t(1-\beta_t) + W_c(1-\beta_c) - M_t\{\log|\beta_t| - \beta_t + 1\} - M_c\{\log|\beta_c| - \beta_c + 1\} \\ &\quad + \frac{k}{2}[(\operatorname{grad}\beta_t)^2 + (\operatorname{grad}\beta_c)^2],\end{aligned} \tag{19}$$

$$\Phi = \Phi(\dot\beta_t, \dot\beta_c; \varepsilon) = \tfrac{1}{2}\{c_t\dot\beta_t^2 + c_c\dot\beta_c^2\} - \dot\beta_c\lambda((\operatorname{tr}[\varepsilon])^-)^2 + I_-(\dot\beta_t, \dot\beta_c), \tag{20}$$

where $I_-(x,y)$ is the indicator function of the set $]-\infty, 0] \times]-\infty, 0]$.

The definitions of the characteristics of the material are identical to those in the model described in Section 3. They are subscripted by t for the extension and c for the contraction. In the expression of Ψ_1, it appears that the contributions of the positive and negative parts of the strain are different. The quadratic function involving the extensions ε^+ is coupled with the damage quantity β_t and the quadratic function involving the contractions ε^- is coupled with the damage quantity β_c.

By using the properties (13), the first constitutive law in eqns (7) gives the stress–strain relation

$$\sigma = \frac{\partial\psi_1}{\partial\varepsilon} = \beta_t\{2\mu\varepsilon^+ + \lambda(\operatorname{tr}[\varepsilon])^+\mathbf{I}_d\} - \beta_c\{2\mu\varepsilon^- + \lambda(\operatorname{tr}[\varepsilon])^-\mathbf{I}_d\}, \tag{21}$$

Let us note that if the material is undamaged ($\beta_t = \beta_c = 1$), relation (21) becomes the classical linear elastic relation.

The equations of evolution obtained with the choices (19) and (20) are

$$\begin{aligned}\operatorname{div}\{\beta_t(2\mu\varepsilon^+ + \lambda(\operatorname{tr}[\varepsilon])^+\mathbf{I}_d) - \beta_c(2\mu\varepsilon^- + \lambda(\operatorname{tr}[\varepsilon])^-\mathbf{I}_d)\} + \mathbf{f} = 0, &\quad \text{in } \Omega, \\ \sigma\cdot\mathbf{n} = \mathbf{F}, &\quad \text{in } \partial\Omega,\end{aligned} \tag{22}$$

$$c_t\dot\beta_t - k\Delta\beta_t + B_t^{reac} + \partial I_-(\dot\beta_t) \ni -\tfrac{1}{2}\{2\mu \operatorname{tr}[\varepsilon^+ \cdot \varepsilon^+] + \lambda((\operatorname{tr}[\varepsilon])^+)^2\} + W_t + M_t\left(\frac{1-\beta_t}{\beta_t}\right), \quad \text{in } \Omega,$$

$$k\frac{\partial\beta_t}{\partial\mathbf{n}} = 0, \quad \text{in } \partial\Omega, \qquad \beta_t(x,0) = \beta_{t_0}(x), \quad \text{in } \Omega,$$

$$c_c\dot{\beta}_c - k\Delta\beta_c + B_c^{\text{reac}} + \partial I_-(\dot{\beta}_c) \ni -\tfrac{1}{2}\{2\mu\,\text{tr}\,[\varepsilon^- \cdot \varepsilon^-]\} + W_c + M_c\left(\frac{1-\beta_c}{\beta_c}\right), \quad \text{in }\Omega, \tag{23}$$

$$k\frac{\partial\beta_c}{\partial\mathbf{n}} = 0, \quad \text{in }\partial\Omega, \quad \beta_c(x,0) = \beta_{c_0}(x), \quad \text{in }\Omega,$$

where the vector $(B_t^{\text{reac}}(\mathbf{x},t), B_c^{\text{reac}}(\mathbf{x},t)) \in \partial I_C(\beta_t(\mathbf{x},t), \beta_c(\mathbf{x},t))$ is a normal vector to the triangle C at the point (β_t, β_c) (see Fig. 1). We can check that these equations give only one velocity $(\dot{\beta}_t, \dot{\beta}_c)$ function of ε and (β_t, β_c) such that $\dot{\beta}_t \leqslant 0$ and $\dot{\beta}_c \leqslant 0$ and (β_t, β_c) remains in the triangle.

A loading history is shown in Fig. 2. One can see qualitatively the restoration of stiffness after damaging in tension (this is the unilateral phenomenon). One can also see that damage in compression is definitive. In other words, when the material is largely damaged in compression, its stiffness cannot be restored when going from compression to traction because it is crushed. This property is a consequence of the choice made for the internal constraint (17) on the damage quantities β_t and β_c.

For the example of concrete, we choose the material characteristics from experimental results (Mazars and Bazant, 1988). In Fig. 3, one can see this time quantitatively, the restoration of stiffness when going from tension to compression. The characteristics of the material used are: $E = 37$ GPa, $\nu = 0.2$, $c_t = 0.002$ MPa · s, $c_c = 0.5$ MPa · s, $W_t = 1 \times 10^{-4}$ MPa, $W_c = 0.7 \times 10^{-2}$ MPa, $M_t = 0.25 \times 10^{-3}$ MPa and $M_c = 0.3 \times 10^{-1}$ MPa.

We note the very different strengths in tension (about 2×10^3 kN m^{-2}) and in compression (about 30×10^3 kN m^{-2}). They are in agreement with the usual values obtained from experiments [see for example Mazars and Bazant (1988)].

5. EXAMPLES OF DAMAGE OF CONCRETE STRUCTURES

In this section we give examples based on the one-damage-quantity model described in Section 3. The loadings are monotonic and do not change sign. Two examples of damage of structures are investigated. The specimens are analysed as two-dimensional. Plane strain is assumed.

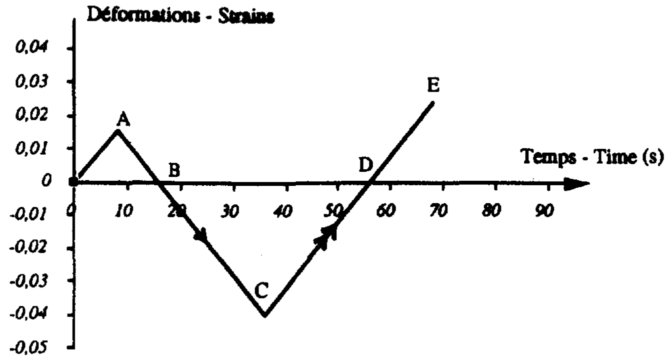
5.1. *First example*

In concrete, most microcracks start from an uncracked surface and grow through the depth of the specimen. Thus, damage mechanics, when applied to concrete, should be able to predict the formation of damage in a specimen which is not notched or precracked. They must also predict the influence of the imposed deformation and the damage growth (Hillerborg, 1983).

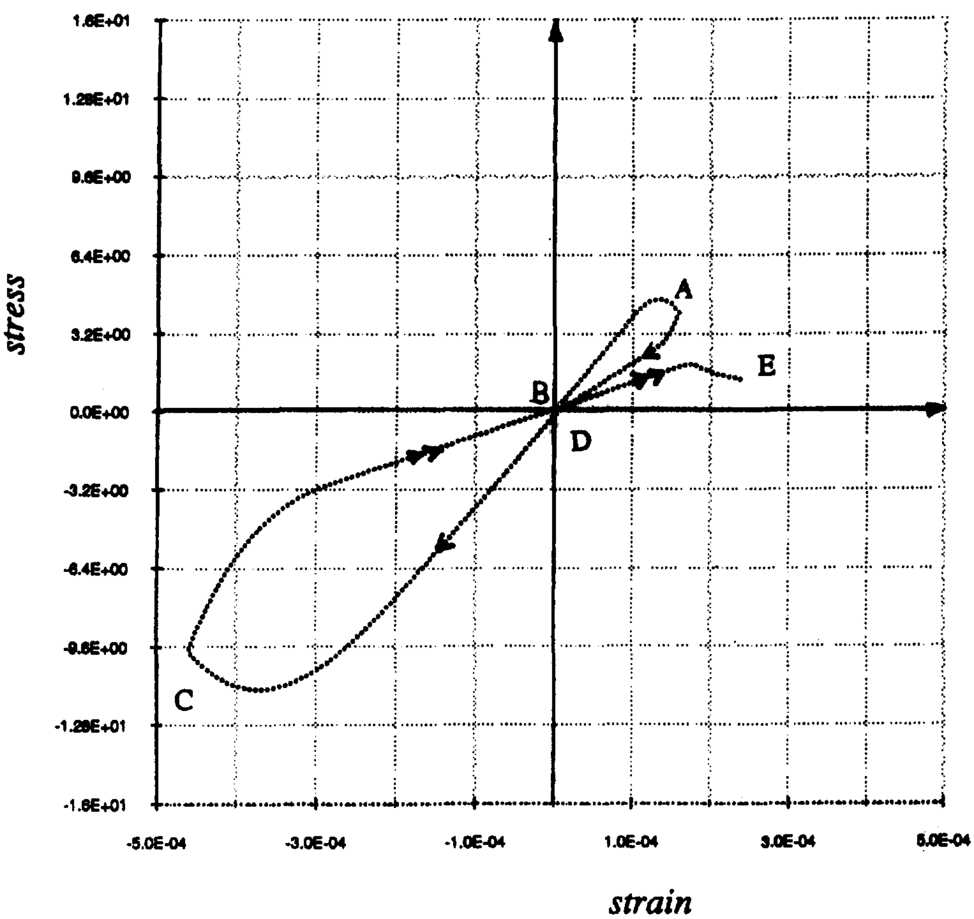
For this purpose, two bending tests under imposed displacements on two identical beams without notch are analysed. The first is a three-point bending test and the second one a four-point bending test. The solicitations, the geometry and the mesh discretization used for both the calculations are shown in Fig. 4.

The mechanical characteristics are $E = 27{,}000$ MPa (Young's modulus) and $\nu = 0.2$ (Poisson's ratio). For the model we use $W = 0.5 \times 10^{-4}$ MPa, $M = 0.25 \times 10^{-3}$ MPa, $c = 0.001$ MPa · s and the factor of influence of damage $k = 0.2$ MPa · mm^2.

A methodology to determine the different parameters of the model from experiments is as follows. From a tensile test performed at slow loading velocity to remain in a quasi-static situation, the first damage threshold W (expressed in terms of volumetric energy) is the strain energy of the material for which the stiffness begins to decrease. For example, if this non-linearity begins at the strain ε_0, $W = E\varepsilon_0^2/2$ (assuming the deformation to be uniaxial). The viscosity parameter of damage c can be identified by performing experiments at different loading velocities (small velocities to remain in a quasi-static situation). The parameter M allows us to describe the softening branch (with small values) of the stress–strain behaviour of the material (concrete for instance) and the non-softening behaviour (with large values) [for instance, a ceramic–ceramic composite (Gasser and Nedjar (1991)].



(*a*)



(*b*)

Fig. 2. (a) Loading history. (b) Constitutive law in tension–compression.

With these three parameters, we get the uniaxial stress–strain behaviour of the material (see for example the behaviour of the concrete plotted in Fig. 11).

The influence of damage factor k results from non-homogeneous loading tests, bending tests for example. It controls the dimension of the damaged zones. Practical values of k between 0.1 and 0.5 $\mathrm{MPa \cdot mm^2}$ give good results in many circumstances (Frémond and Nedjar, 1993a, b, 1994).

The force versus displacement curves of the two tests are superposed and plotted in Fig. 5. It is important to note that these curves show no snap-back instability (Carpinteri and Valente, 1988).

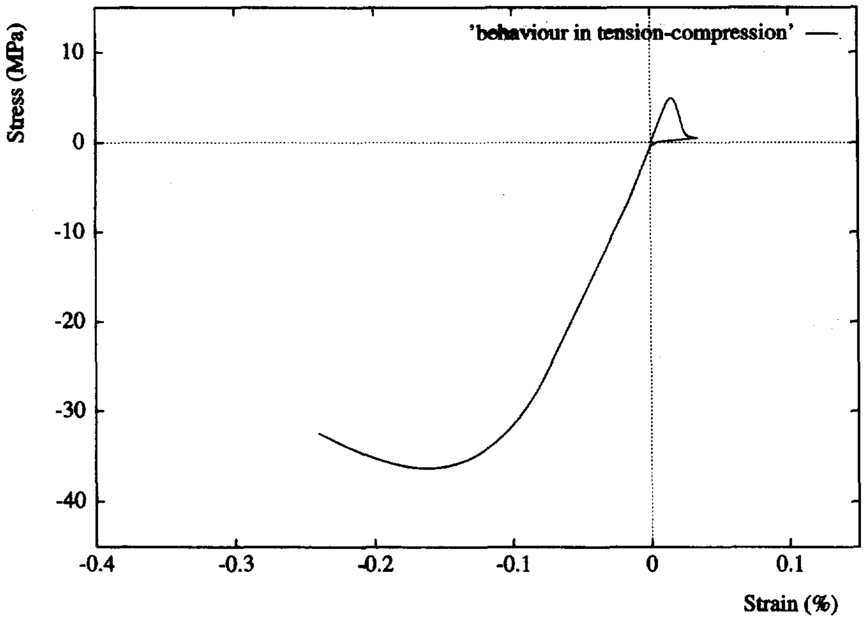


Fig. 3. Concrete constitutive law in tension–compression.

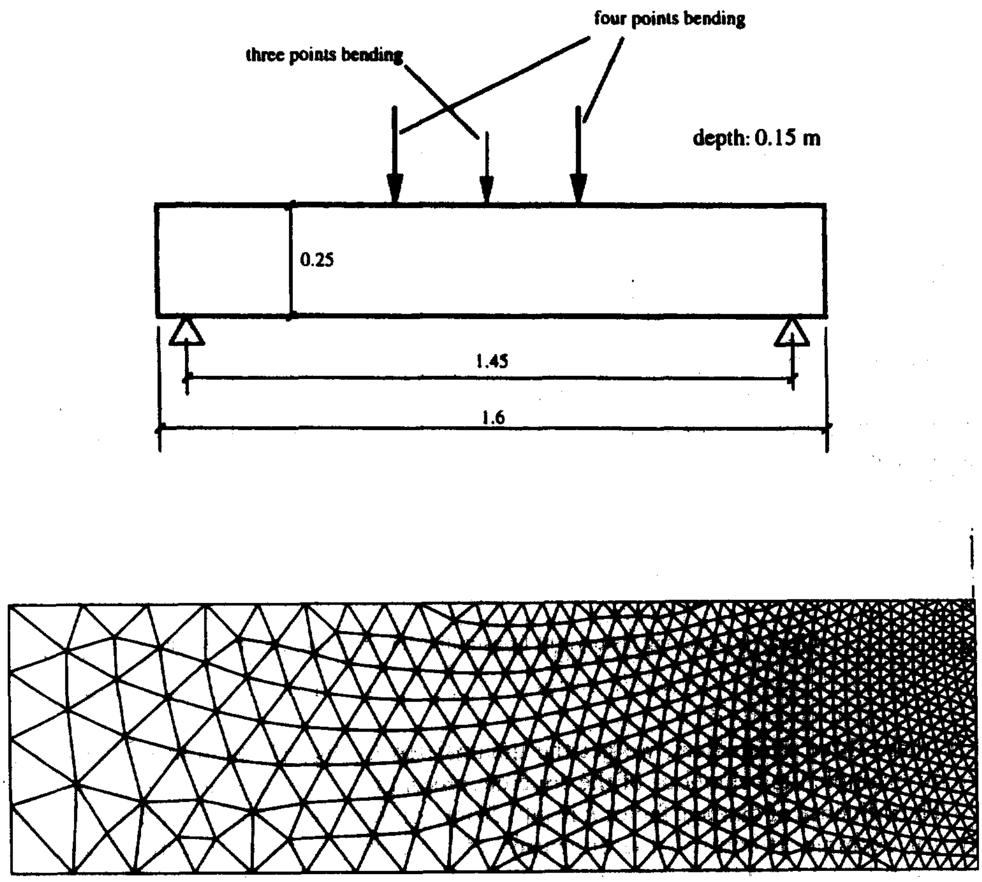


Fig. 4. Three- and four-point bending. Geometry and meshes of half of the specimen used for computation.

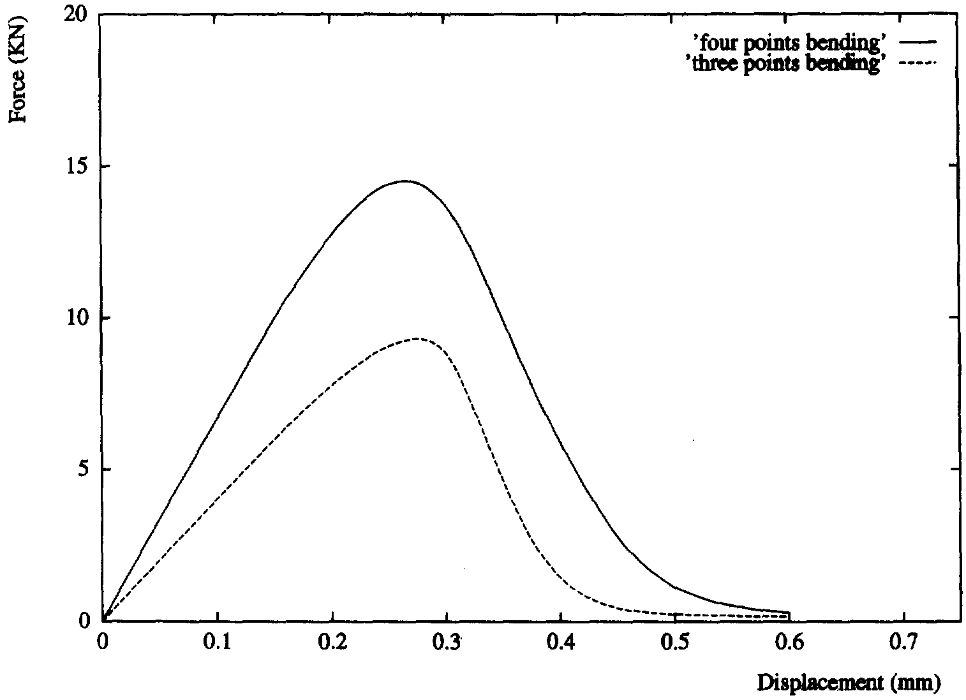


Fig. 5. Force versus displacement curves for the three- and four-point bending tests.

It is interesting to compare the formation and the growth of damage for the two tests predicted by the model. For this purpose, Fig. 6 shows the juxtaposition of the damage fields at three loading steps for each test. The first corresponds to the beginning of the formation of the damage, the second to the ultimate load withstood by each beam (at the peak; see Fig. 5) and the third to a post-peak situation just before the beams are completely damaged through their depths.

5.2. *Second example*

This example intends to show the lack of mesh sensivity of the predictive theory. The same physical problem is solved with different meshes to see whether or not the solutions converge. The computations are carried out for the notched concrete plate shown in Fig. 7 with three meshes.

The material characteristics of the concrete used are $E = 33$ GPa and $\nu = 0.2$; for the characteristics of the model we have $W = 0.75 \times 10^{-4}$ MPa, $M = 0.2 \times 10^{-3}$ MPa, $c = 0.005$ MPa·s and $k = 0.2$ MPa·mm^2.

The three "opening force F versus the aperture a of the notch" curves corresponding to the three meshes are plotted together with experimental results (Mazars and Walter, 1980) in Fig. 8. One can note the objectivity of these results. There is no mesh sensitivity with regard to the global behaviour of the structure. One can observe the good prediction of the actual ultimate load.

To illustrate the lack of mesh sensitivity of the damage process, Fig. 9 shows the damage field predicted by the model for the three meshes at the aperture of the notch $a = 0.22$ mm (see Fig. 8).

5.3. *Comments*

These two examples illustrate the possibilities of the damage model issued from the theory and its coherence from the mechanical point of view. They do not exhibit mesh sensitivity and are able to predict the formation and growth of damage in concrete under multiaxial loading.

6. GRADIENT OF DAMAGE AND STRUCTURAL SIZE EFFECT

The important size effect in concrete has been the subject of many experimental investigations in the laboratory and has been the subject of many theoretical studies [see for instance Bazant and Ozbolt (1990), Hillerborg (1983) and Saouridis and Mazars (1988)].

One of the fundamental results of these studies is that it is observed that the nominal stress (Bazant and Ozbolt, 1990; Mazars *et al.*, 1991; Biolzi *et al.*, 1994) decreases when increasing the scale of the tested specimens. We can prove that this property is predicted by the models based on the theory described in Section 2. We emphasize this result with numerical examples compared to experiments.

The numerical examples we show correspond to three-point bending experiments on notched beams with three size ratios: 1, 2 and 4. They are related to experimental results given in Bazant and Ozbolt (1990). Figure 10 shows half of the small specimen with the mesh used for computation. The two other specimens and their meshes are similar with ratios 2 and 4. The plane strain state is assumed for every computation.

The depth of the notch is always one-sixth of the depth of the beam. The depth of the smallest beam is $d = 7.62 \times 10^{-2}$ m and its thickness is $b = 3.81 \times 10^{-2}$ m.

The characteristics of the material (concrete) are as in experiments: $E = 35{,}000$ MPa and $\nu = 0.18$. The characteristics used for the model are $c = 0.002$ MPa·s, $W = 0.42 \times 10^{-4}$ MPa and $M = 0.25 \times 10^{-3}$ MPa. We choose the value of the factor of influence of damage $k = 0.1$ MPa·mm^2. For concrete, such a value gives good results in many other circumstances (Frémond and Nedjar, 1993a,b, 1994).

With these characteristics, the tensile strength of the concrete, which does not depend on k, is $f_t = 3.27$ MPa, as shown in Fig. 11. This concrete strength in tension agrees with the experimental strength given in Bazant and Ozbolt (1990).

The numerical results are shown in Fig. 12. As for the examples of Section 5, the "forces versus displacements" curves show no snap-back instability (Carpinteri and Valente, 1988).

The ultimate loads P are analysed for each test and the values of the nominal stresses are given by the expression $\sigma_N = P_\delta/bd\delta^2$ as in Bazant and Ozbolt (1990) (with $\delta = 1$ for the smallest beam, $\delta = 2$ for the middle beam and $\delta = 4$ for the largest beam), where P_δ is the ultimate load supported by the structure of size ratio δ (see Fig. 12). These nominal stresses are plotted in Fig. 13, where one can see the computed and experimental nominal stresses versus the size. A good correlation with the experimental test data is observed: the nominal stress decreases when the size of the beam increases.

7. A FATIGUE DAMAGE MODEL

Two kinds of damage processes are usually investigated: brittle damage and fatigue damage. The models studied previously are brittle damage models. Fatigue damage is mainly produced by an accumulation of damage during a high number of loading cycles (Marigo, 1985; Lemaitre, 1992; Papa, 1993). We propose in this section a simple extension of the damage theories studied in Section 3 to model the fatigue damage.

The basic idea is to consider that the threshold of damage decreases with the number of cycles N. The damage threshold W is then such that $W = W(N)$. In this case, the number N is considered to be a global state quantity of the structure. The free energy Ψ_1 will also depend on N, as well as the usual quantities:

$$\Psi_1 = \Psi_1(\varepsilon, \beta, \operatorname{grad}\beta, N). \tag{24}$$

By using continuum thermodynamics, the Clausius–Duhem inequality is satisfied if the following condition is satisfied:

$$\frac{\Delta\Psi_1}{\Delta N} \leqslant 0. \tag{25}$$

The constitutive relations (7) are still valid. The condition (25) is in agreement with

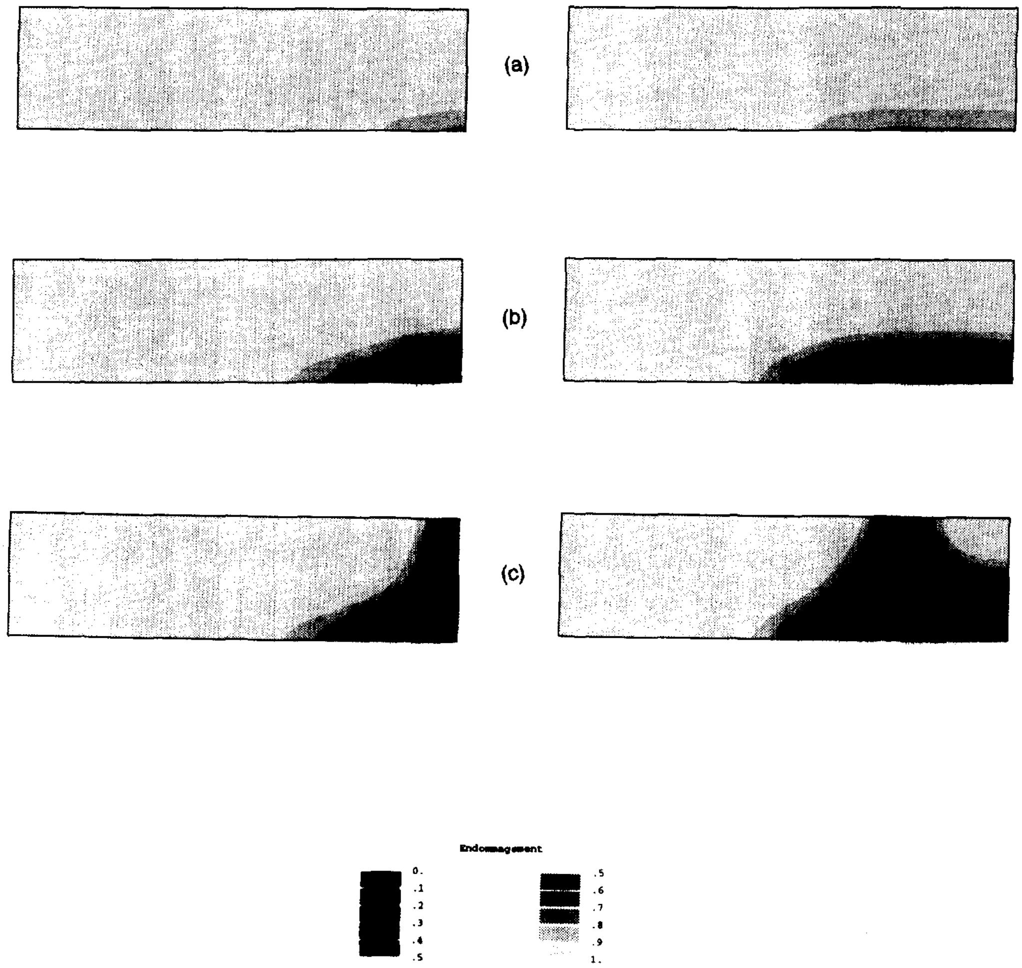


Fig. 6. Damage field at displacements 0.2 mm (a), 0.3 mm (b) and 0.5 mm (c).

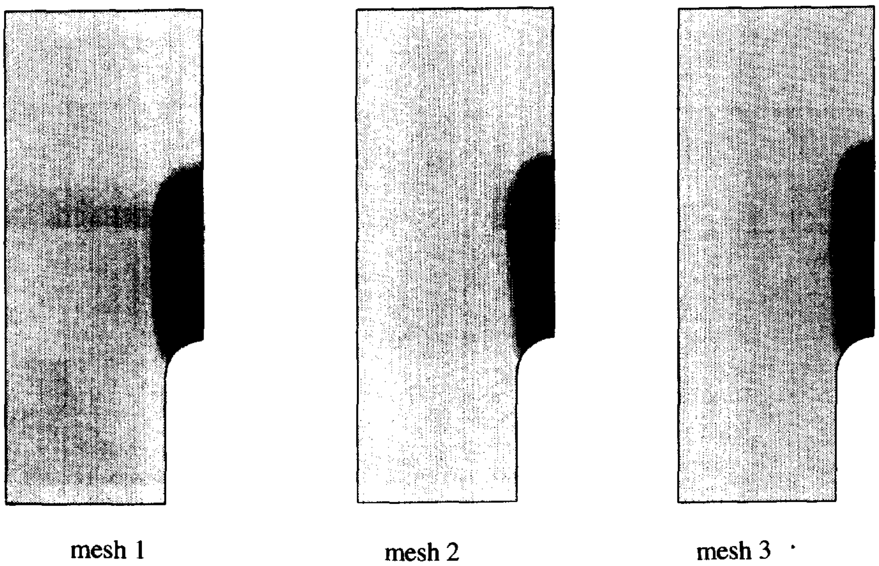


Fig. 9. Damage fields for the three meshes. The aperture of the notch is 0.22×10^{-3} m.

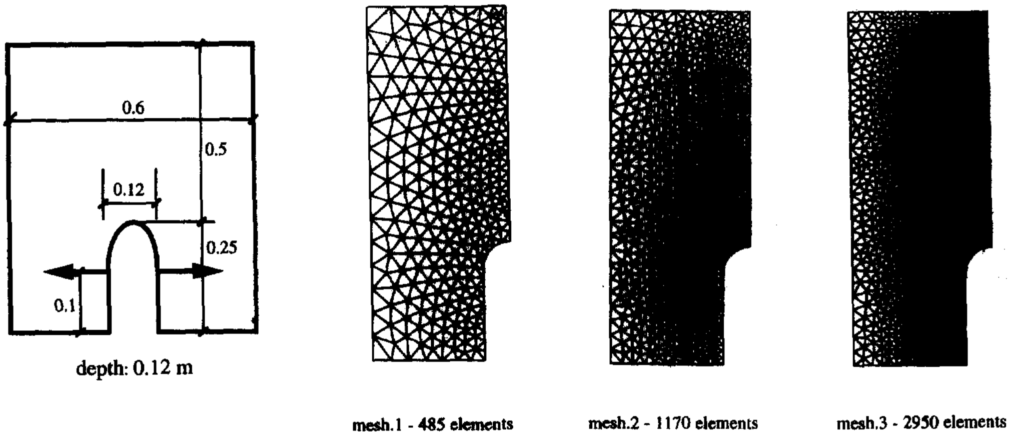


Fig. 7. The plate and the three meshes.

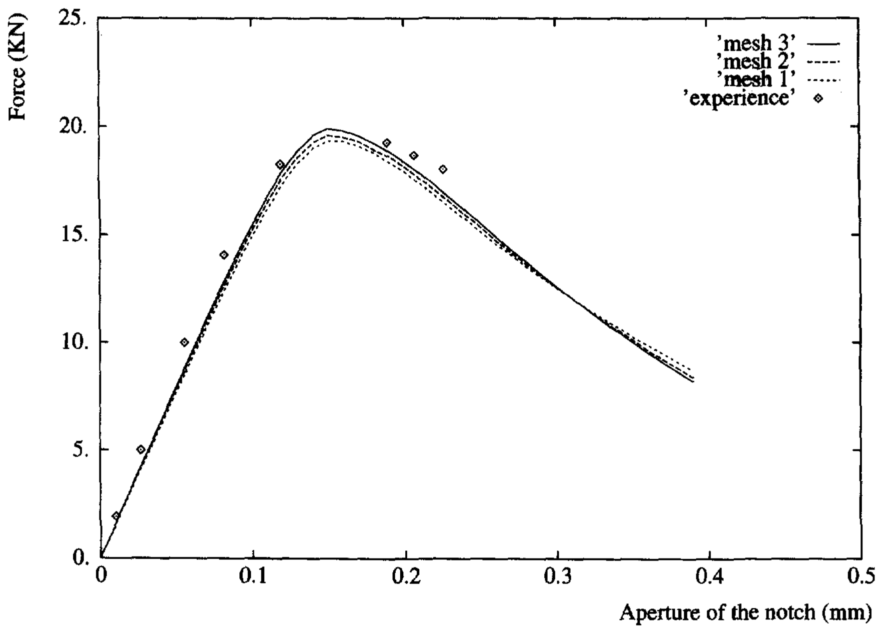


Fig. 8. The opening force versus the aperture of the notch. Experimental and numerical results for the three meshes.

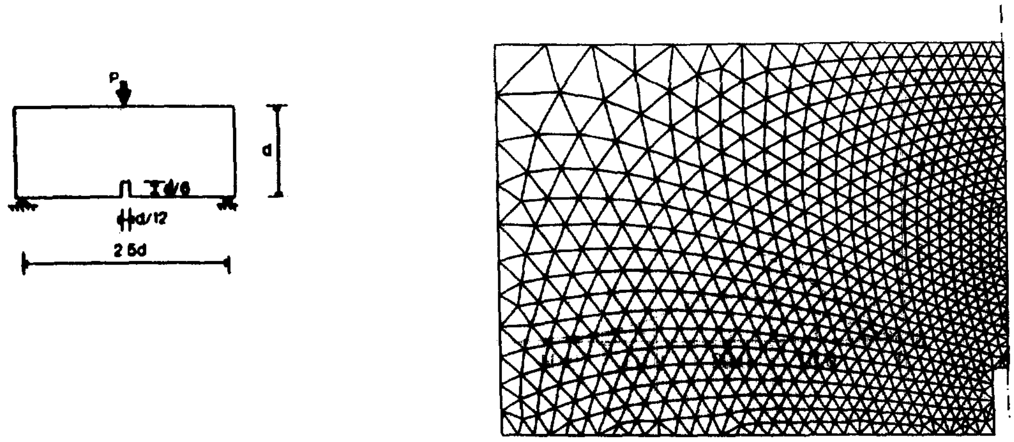


Fig. 10. Three-point bending. Meshes of half of the small specimen.

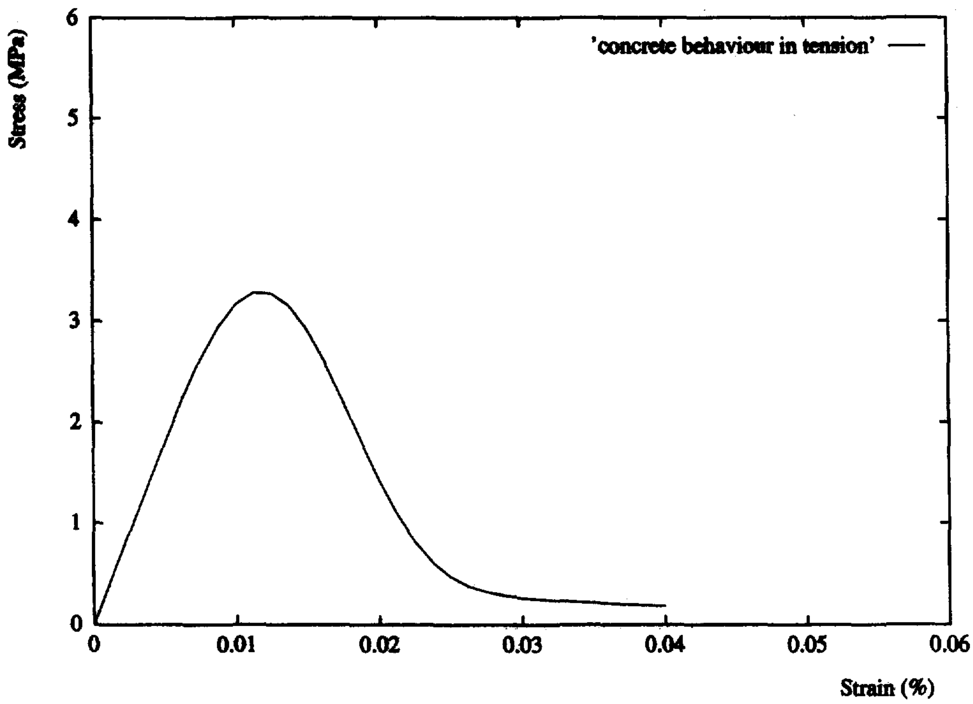


Fig. 11. Concrete behaviour in tension.

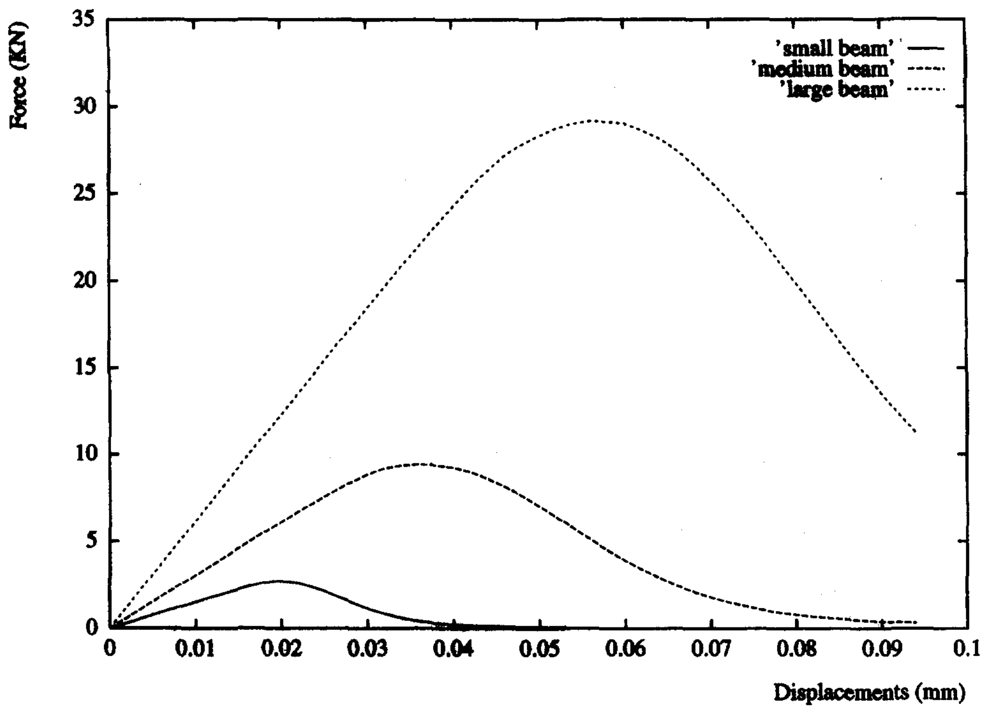


Fig. 12. Three-point bending tests, force versus displacement.

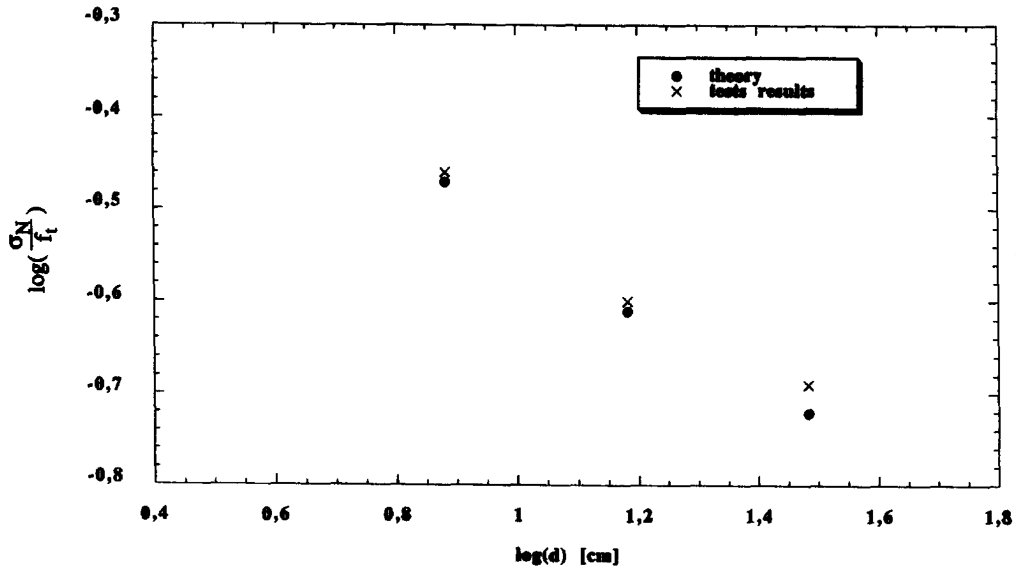


Fig. 13. Nominal stress σ_N versus the size d of the three specimens.

the property mentioned above, i.e. the threshold decreases with the number of applied cycles. Such a condition is satisfied by the following general expression of the free energy:

$$\Psi_1 = \tfrac{1}{2}\beta\{2\mu\,\mathrm{tr}\,[\varepsilon\varepsilon] + \lambda(\mathrm{tr}\,[\varepsilon])^2\} + W(N)(1-\beta) - M(\log(|\beta|) - \beta + 1) + \frac{k}{2}(\mathrm{grad}\,\beta)^2, \tag{26}$$

with

$$\frac{\Delta W}{\Delta N} \leqslant 0.$$

The pseudo-potential of dissipation is the one given in eqn (12) and the constitutive relations are those given by eqns (14).

The function $W(N)$ has to be such that $W(0) = W_0$, the initial threshold, and $W(\infty) = 0$. A simple choice of this function can be given by

$$W(N) = W_0\left(1 - \frac{N}{N_R}\right) \quad \text{if } N \leqslant N_R,$$
$$W(N) = 0. \qquad \text{if } N > N_R. \tag{27}$$

N_R can be interpreted as the number of cycles to achieve complete damage. We note that if N_R tends to $+\infty$, we get the brittle damage model described in section 3.

A qualitative example of fatigue damage behaviour using the threshold function (27) is plotted in Fig. 14. The value of the quantity N_R chosen is $N_R = 20$. This behaviour is the response to the loading history given in Fig. 15, where strain controlled loading is assumed.

8. CONCLUSION

A new formulation of damage based on the principle of virtual power is developed. It introduces the gradient of damage, which physically accounts for the influence of damage at a material point on the damage of its neighbourhood. It appears that this formulation is coherent from the mechanical, mathematical and numerical points of view.

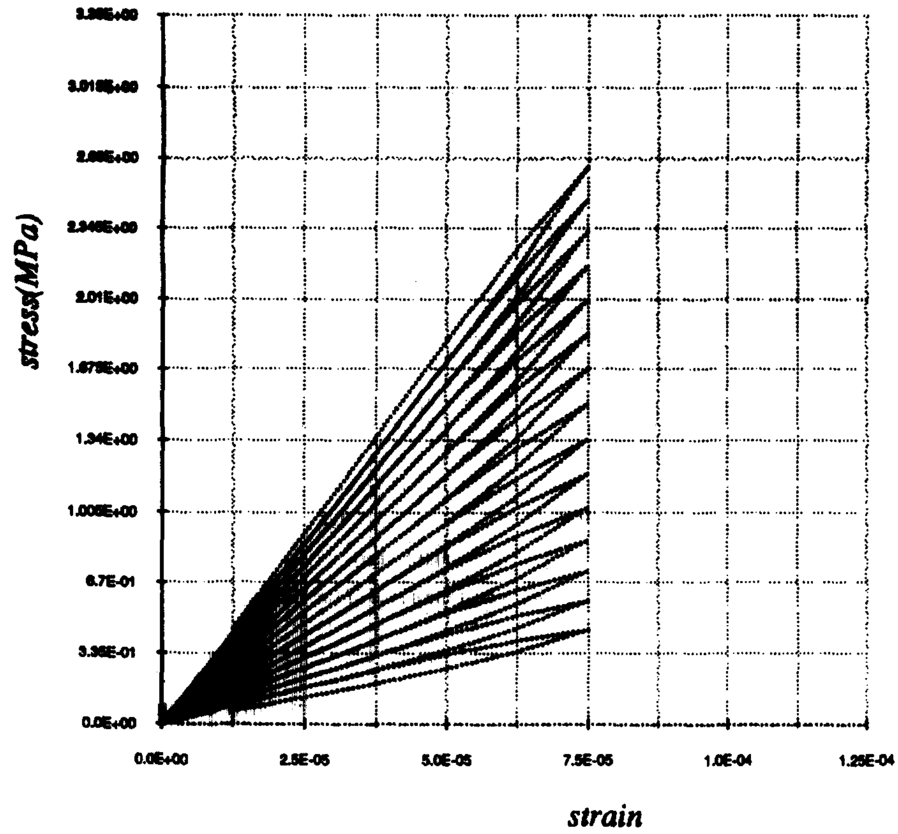


Fig. 14. Fatigue damage behaviour. Qualitative example. $E = 35{,}000$ MPa, $\nu = 0.18$, $c = 2 \times 10^{-3}$ MPa·s, $W = 0.42 \times 10^{-4}$ MPa, $M = 0.25 \times 10^{-3}$ MPa and $N_R = 20$.

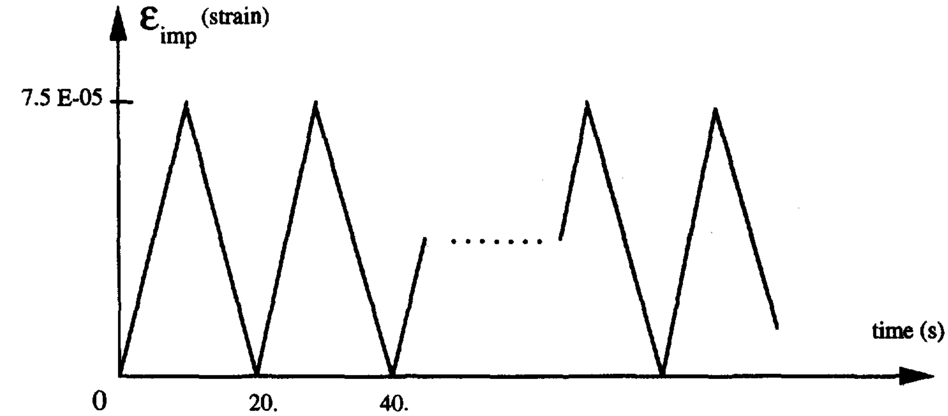


Fig. 15. Loading history. Strain controlled loading.

It is shown that the resulting theories do not exhibit mesh sensitivity. They are able to predict correctly the concrete behaviour in multiaxial loading situations with a good prediction of the very important structural size effect.

The unilateral phenomenon is described by using two damage quantities such that damage in compression produces damage in tension, and the reverse is not true. Finally, an extension to predict the fatigue damage behaviour is proposed.

REFERENCES

Bazant, Z. P. and Ozbolt, J. (1990). Non local microplane model for fracture, damage and size effect in structures. *J. Engng Mech. ASCE* **116**, 2485–2505.

Biolzi, L., Gambarova, P. G., Rosati, G. P. and Schumm, C. (1994). On fracture and size effect in concrete beams. In *Computer Modelling of Concrete Structures* (Edited by H. Mang, N. Bicanic and R. De Borst), *Proc. Euro-C* 1994, *Int. Conf.*, Austria, Vol. 1, pp. 53–62.

Brauner, C. M., Nicolaenco, B. and Frémond, M. (1986). Homographic approximations of free boundary problems characterized by variational inequalities. *Sci. Comput. Adv. Math. Suppl. Stud.* **10**, 119–151.

Carpinteri, A. and Valente, S. (1988). Size-scale transition from ductile to brittle failure: a dimensional analysis approach. In *Cracking and Damage, Strain Localization and Size Effect* (Edited by J. Mazars and Z. P. Bazant), pp. 477–490. Elsevier, Amsterdam.

Chaboche, J. L. (1992). Une nouvelle condition unilatérale pour décrire le comportement des matériaux avec dommage anisotrope. *C. R. Acad. Sci. Paris* **314**, ser. II, 1395–1401.

Costa Mattos, H., Frémond, M. and Namiya, E. N. (1992). A simple model of the mechanical behaviour of ceramic like material. *Int. J. Solids Structures* **29**, 3185–3200.

Frémond, M. (1987). Adhérence des solides. *J. Méc. Théor. Appl.* **6**(3), 383–407.

Frémond, M. (1990). Sur l'inégalité de Clausius–Duhem. *C. R. Acad. Sci. Paris* **311**, ser. II, 757–762.

Frémond, M. and Nedjar, B. (1993a). Endommagement et principe des puissances virtuelles. *C. R. Acad. Sci. Paris* **317**(7), ser. II, 857–864.

Frémond, M. and Nedjar, B. (1993b). Damage and gradient of damage. The unilateral phenomenon. *Proc. SMiRT*12 (*12th Structural Mechanics in Reactor Technology*) (Edited by F. Kussmaul), Vol. H, pp. 375–380. North-Holland, Amsterdam.

Frémond, M. and Nedjar, B. (1994). Damage of concrete, the unilateral phenomenon. *J. Nucl. Engng Design* (in press).

Gasser, A. and Nedjar, B. (1991). Une première méso-modélisation d'un composite à matrice céramique SiC–SiC. Mémoire de D.E.A., Laboratoire de Mécanique et Technologie, Ecole Normale Supérieur de Cachan.

Germain, P., Nguyen, Q. S. and Suquet, P. (1983). Continuum thermodynamics. *J. Appl. Mech., ASME* **50**, 1010–1021.

Hillerborg, A. (1983). Analysis of a single crack. In *Fracture Mechanics of Concrete* (Edited by F. H. Wittman), pp. 223–249. Elsevier, Amsterdam.

Ladevèze, P. (1983). Sur une théorie de l'endommagement anisotrope. Internal report no. 34, Laboratoire de Mécanique et Technologie, Ecole Normale Supérieur de Cachan.

Lemaitre, J. (1992). *A Course on Damage Mechanics*. Springer, Berlin.

Lemaitre, J. and Chaboche, J. L. (1988). *Mécanique des Matériaux Solides*, 2nd edn. Dunod, Paris.

Marigo, J. J. (1985). Modelling of brittle and fatigue damage for elastic material by growth of microvoids. *Engng Fracture Mech.* **21**, 861–874.

Maugin, G. A. (1990). Internal variables and dissipative structures. *J. Non-Equilib. Thermodyn.* **15**, 173–192.

Mazars, J. and Bazant, Z. P. (Eds) (1988). *Cracking and Damage, Strain Localization and Size Effect*. Elsevier, Amsterdam.

Mazars, J., Pijaudier-Cabot, G. and Saouridis, C. (1991). Size effect and continuous damage in cementitious materials. *Int. J. Fracture* **51**, 159–173.

Mazars, J. and Walter, D. (1980). Endommagement mécanique du béton. Délégation Générale à la Recherche Scientifique et Technique, no. 78.7.2697 and 78.7.2698, France.

Moreau, J. J. (1966). Fonctionnelles convexes. Séminaire sur les équations aux dérivées partielles, Collège de France, Paris.

Moreau, J. J. (1970). Sur les lois de Frottement, de viscosité et de plasticité. *C. R. Acad. Sci. Paris* **271**, 608–611.

Nedjar, B. (1993). A first gradient theory for damage. *Proc. 1st Forum of Young European Researchers*, Liège, Belgum, pp. 199–204.

Papa, E. (1993). A damage model for concrete subjected to fatigue loading. *Eur. J. Mech. A/Solids* **12**, 429–440.

Pijaudier-Cabot, G. and Bazant, Z. P. (1987). Non-local damage theory. *J. Engng Mech., ASCE* **113**, 1512–1533.

Pijaudier-Cabot, G., La Borderie, C. and Fichant, S. (1994). Damage mechanics for concrete modelling: application and comparison with plasticity and fracture mechanics. *Proc. Euro-C 1994, Int. Conf.*, Austria (Edited by H. Mang, N. Bicanic and R. De Boorst), Vol. I, pp. 17–36.

Saouridis, C. and Mazars, J. (1988). A multiscale approach to distributed damage and its usefulness for capturing structural size effect. In *Cracking and Damage* (Edited by J. Mazars and Z. P. Bazant), pp. 391–403. Elsevier, Amsterdam.